TRANSFORMATIONS

Ex 1: $f_X(x) = \frac{1}{2}$, $0 < x < 2$ (Uniform distn.)

let $Y = X^2$. What is the pdf of Y?

$P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y})$

$= \int_0^{\sqrt{y}} f_X(x)\,dx = \left(\frac{1}{2}x\right)\Big|_0^{\sqrt{y}} = \frac{\sqrt{y}}{2}$, $0 \le y < 4$

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ \sqrt{y}/2, & \text{if } 0 \le y < 4 \\ 1, & \text{if } y \ge 4 \end{cases}$$

$f_Y(y) = \frac{1}{4\sqrt{y}}$, $0 < y < 2$ and zero elsewhere.

Note that $Y = X^2$ is a monotone (and increasing) fnc. in the interval $0 < x < 2$.

It would not be a monotone fnc. if we allowed, for instance, $-2 < x < 2$.

** The technique used here is called the "distribution function technique".

See the following theorem, which formalizes what we did above.

Theorem: Let $f_X(x)$ be the density of r.v. X and let $Y = u(X)$, where $u(\cdot)$ is a monotonic differentiable function. Then, the density of Y is given by:

(Theorem 3.6.1. from Amemiya)

$$f_Y(y) = f_X(u^{-1}(y)) \cdot \left|\frac{du^{-1}}{dy}\right| ,$$

where u^{-1} is the inverse function of u.

$\longrightarrow$

<u>Proof</u>: $P(Y \leq y) = P(u(X) \leq y)$

a) If $u(\cdot)$ is increasing, then $= P(X \leq u^{-1}(y))$

Therefore $F_Y(y) = F_X(u^{-1}(y))$

Differentiate both sides (with respect to y) to find

$$f_Y(y) = f_X(u^{-1}(y)) \cdot \frac{d\,u^{-1}(y)}{dy}$$

b) If $u(\cdot)$ is decreasing, then

$$P(u(X) \leq y) = P(X > u^{-1}(y)) = 1 - F_X(u^{-1}(y))$$

Again, diff. both sides with respect to y,

$$f_Y(y) = -f_X(u^{-1}(y)) \cdot \frac{d u^{-1}(y)}{dy}$$

$$\Rightarrow f_Y(y) = f_X(u^{-1}(y)) \cdot \left| \frac{d u^{-1}(y)}{dy} \right| .$$ Established.

** This technique is called the "change-of variables technique".

The distn. fnc. technique is more lengthy since it does not use the short-cut provided by the theorem that we proved above. However, its advantage is that it is more fundamental.

Keep in mind that the "change-of-variables tech." (as presented in Thm. 3.6.1) is suitable only for monotone transformations, but the "distn. fnc. tech." is more generally applicable.

<u>EX</u> 2: $f_X(x) = \frac{1}{2}$, $-1 < x < 1$. Let $Y = X^2$.

Try and see how a careless application of the change-of-variables tech. yields a wrong answer.

Use the distn. fnc. tech. to get the correct answer.

Next, we will see the application of the change-of-variables tech. to the multivariate case.

Theorem 3.6.3 (Amemiya) : Let $f_{X_1,X_2}(x_1,x_2)$ be the joint density of random variables X_1 and X_2.

Let Y_1 and Y_2 be defined by a linear transformation:

$$Y_1 = a_{11}X_1 + a_{12}X_2 \,,$$
$$Y_2 = a_{21}X_1 + a_{22}X_2 \,.$$

Suppose $a_{11}a_{22} - a_{12}a_{21} \neq 0$ so that we can express X_1, X_2 as:

$$X_1 = b_{11}Y_1 + b_{12}Y_2$$
$$X_2 = b_{21}Y_1 + b_{22}Y_2$$

Then, the joint density $f_{Y_1,Y_2}(y_1,y_2)$ is given by

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)\cdot\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}$$
$$= \frac{f_{X_1,X_2}(b_{11}y_1+b_{12}y_2\,,\ b_{21}y_1+b_{22}y_2)}{|a_{11}a_{22}-a_{12}a_{21}|}\,.$$

In Thm 3.6.3., $|a_{11}a_{22}-a_{12}a_{21}|$ is the absolute value of the determinant of the 2x2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Ex 3: $f_{X_1,X_2}(x_1,x_2) = 4x_1x_2$, $0 \le x_1 \le 1$, $0 \le x_2 \le 1$.

$Y_1 = X_1 + 2X_2$, $Y_2 = X_1 - X_2$. What is the pdf $f_{Y_1,Y_2}(y_1,y_2)$?

We can show that $X_1 = \frac{1}{3}Y_1 + \frac{2}{3}Y_2$, $X_2 = \frac{1}{3}Y_1 - \frac{1}{3}Y_2$

$$f_{Y_1,Y_2}(y_1,y_2) = 4\left(\tfrac{1}{3}y_1 + \tfrac{2}{3}y_2\right)\left(\tfrac{1}{3}y_1 - \tfrac{1}{3}y_2\right)\cdot\frac{1}{3}$$

$\det\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = -3$, $|-3| = 3$

{See the note on page (4).}

Another ex. of the use of the change-of-variables technique.

Ex 4: $f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} & , y_1>0, y_2>0 \\ 0 & , \text{elsewhere.} \end{cases}$

Let $U = Y_2/Y_1$. Find the pdf. of U.

$U = Y_2/Y_1$, Set $V = Y_1$. ⤳ Use the denominator to select the 2nd function.

Then, $Y_1 = V$
$Y_2 = UY_1 = UV$

The "Jacobian" of this transformation?

$$J = \begin{vmatrix} \partial Y_1/\partial U & \partial Y_1/\partial V \\ \partial Y_2/\partial U & \partial Y_2/\partial V \end{vmatrix}$$

$$J = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = -v$$

$$\Rightarrow |J| = v$$

$$f_{U,V}(u,v) = f_{Y_1,Y_2}(y_1,y_2)\cdot|J|$$
$$= \frac{1}{8} y_1 e^{-(y_1+y_2)/2}\cdot v$$
$$= \frac{1}{8} v^2 e^{-\frac{v}{2}(1+u)}$$

$f_U(u) = ?$

$$f_U(u) = \int_0^\infty \frac{1}{8} v^2 e^{-\frac{v}{2}(1+u)}\,dv$$

Use integration by parts $\Rightarrow$ $f_U(u) = \dfrac{2}{(1+u)^3}$, $u>0$.

Note on Ex 3: Alternative solution: (Using the method described above.)

$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)\cdot|J|$ ↳ the Jacobian of the transformation.

$$J = \begin{vmatrix} \partial X_1/\partial Y_1 & \partial X_1/\partial Y_2 \\ \partial X_2/\partial Y_1 & \partial X_2/\partial Y_2 \end{vmatrix} = \begin{vmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

$|J| = \frac{1}{3}$. Therefore, $f(y_1,y_2) = 4\left(\frac{1}{3}y_1+\frac{2}{3}y_2\right)\left(\frac{1}{3}y_1-\frac{1}{3}y_2\right)\cdot\left(\frac{1}{3}\right)$

There is another technique called the "moment generating function technique".

<u>EX</u>: Let $X \sim N(0,1)$ and $Y = 3X+2$

We know that the mgf of normal distn. with mean μ and variance σ^2 is:

$$M(t) = E(e^{tX}) = e^{\mu t + \sigma^2 t^2/2}$$

If we can generate the mgf of the r.v. of Y, we can (perhaps) say the distribution of Y.

$$M_Y(t) = E(e^{tY}) = E\left(e^{t(3X+2)}\right) = E\left(e^{3tX+2t}\right)$$

$$= \underset{\Downarrow}{E\left(e^{3tX}\right)} \cdot \underbrace{E\left(e^{2t}\right)}_{\text{a constant}}$$

the mgf of X, where $3t$ is used instead of t.

We know that $X \sim N(0,1)$, so $M_X(t) = e^{t^2/2}$

$$\Rightarrow E\left(e^{(3t)X}\right) = e^{(3t)^2/2} = e^{9t^2/2}$$

$$M_Y(t) = E\left(e^{(3t)X}\right) \cdot E\left(e^{2t}\right) = e^{9t^2/2} \cdot e^{2t} = e^{2t + 9t^2/2}$$

But this is the mgf of a normally distributed r.v. whose mean is 2 and variance is 9. $(\mu = 2,\ \sigma^2 = 9)$

→ We could have used the distn. fnc. technique instead.

The pdf of $X \sim N(0,1)$ is $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $-\infty < x < \infty$

$$P(Y \le y) = P(3X+2 \le y) = P\left(X \le \frac{y-2}{3}\right) = \int_{-\infty}^{\frac{y-2}{3}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Change x to y. We know $x = \frac{y-2}{3}$, $dx = dy/3$

$$\int_{-\infty}^{x} \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-2}{3}\right)^2} dy$$. Diff. with respect to y

$$\Rightarrow f_Y(y) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-2}{3}\right)^2}$$. But this is the pdf of $Y \sim N(2, 9)$.

Ex : $Y_1 \sim N(0,1)$ $\quad$ Y_1 and Y_2 independent.

$Y_2 \sim N(5,9)$ $\quad$ (This is an important assumption.)

Find the pdf of $U = 3Y_1 + 2Y_2$

$$M_U(t) = E(e^{tU}) = E\left(e^{t3Y_1 + t2Y_2}\right)$$

due to independence $= E\left(e^{t3Y_1}\right) . E\left(e^{t2Y_2}\right)$

$$M_{Y_1}(t) = E(e^{tY_1}) = e^{t^2/2}$$

$$M_{Y_2}(t) = E(e^{tY_2}) = e^{5t + 9t^2/2}$$

$$M_U(t) = E\left(e^{(3t)Y_1}\right) . E\left(e^{(2t)Y_1}\right)$$

$$= e^{9t^2/2} . e^{10t + 36t^2/2} = e^{10t + 45t^2/2}$$

$$\Rightarrow U \sim N(10, 45) \qquad \mu = 10, \sigma^2 = 45$$

We generalize this result in the following theorem on page 7.

Theorem 1:

$X_1, X_2, \ldots, X_n$ independent r.v.'s having normal distributions $N(\mu_1, \sigma_1^2), N(\mu_2, \sigma_2^2), \ldots, N(\mu_n, \sigma_n^2)$.

Let $Y = k_1X_1 + k_2X_2 + \cdots + k_nX_n$ where k are real constants.

Then Y is distributed normally with

mean $= k_1\mu_1 + k_2\mu_2 + \cdots + k_n\mu_n$ and

variance $= k_1^2\sigma_1^2 + k_2^2\sigma_2^2 + \cdots + k_n^2\sigma_n^2$.

In other words, $Y \sim N\left(\sum_1^n k_i\mu_i, \sum_1^n k_i^2\sigma_i^2\right)$

Proof:

Due to independence of the X's, we can express the mgf as the product of several mgf.'s.

$$M_Y(t) = E\left[e^{t(k_1X_1 + k_2X_2 + \cdots + k_nX_n)}\right]$$

$$= E\left(e^{tk_1X_1}\right) \cdot E\left(e^{tk_2X_2}\right) \cdots E\left(e^{tk_nX_n}\right)$$

$$E\left(e^{tX_i}\right) = e^{\mu_i t + \sigma_i^2 t^2/2}$$

$$E\left(e^{tk_iX_i}\right) = E\left(e^{(k_it)X_i}\right) = e^{\mu_i(k_it) + \sigma_i^2(k_it)^2/2}$$

$$M_Y(t) = \prod_{i=1}^n e^{k_i\mu_i t + \sigma_i^2 k_i^2 t^2/2}$$

$$M_Y(t) = e^{t\sum_{i=1}^n k_i\mu_i + (t^2/2)\sum_{i=1}^n \sigma_i^2 k_i^2}$$

This is the mgf of a normal distribution with mean $= \sum_1^n k_i\mu_i$ and variance $= \sum_1^n k_i^2\sigma_i^2$

→ A generalization of Theorem 1 follows.

Theorem 2 :

$X_1, X_2, \ldots, X_n$ independent r.v.'s with mgf $M_i(t)$ $i=1,2,\ldots,n$.

The mgf of $Y = \sum_{i=1}^{n} a_i X_i$ (a's are real constants)

is $M_Y(t) = \prod_{i=1}^{n} M_i(a_i t)$

Proof : Independence $\Rightarrow$

$$M_Y(t) = E(e^{tY}) = E\left(e^{t(a_1X_1 + a_2X_2 + \cdots + a_nX_n)}\right)$$
$$= E(e^{a_1 t X_1}) \cdot E(e^{a_2 t X_2}) \cdots E(e^{a_n t X_n})$$

$M_i(t)$: The mgf of X_i.

$M_i(t) = E(e^{tX_i}) \quad \Rightarrow \quad E(e^{a_i t X_i}) = M_i(a_i t)$

$$\Rightarrow M_Y(t) = M_1(a_1 t) \cdot M_2(a_2 t) \cdots M_n(a_n t)$$
$$= \prod_{i=1}^{n} M_i(a_i t)$$

Corollary : $X_1, X_2, \ldots, X_n$ are observations of a random sample from a distn. with mgf $M(t)$.

a) The mgf of $Y = \sum_{i=1}^{n} X_i$ is $M_Y(t) = \prod_{i=1}^{n} M(t) = [M(t)]^n$.

b) The mgf of $\bar{X} = \sum_{i=1}^{n} (1/n) X_i$ is

$$M_{\bar{X}}(t) = \prod_{i=1}^{n} M\left(\frac{t}{n}\right) = \left[M\left(\frac{t}{n}\right)\right]^n$$

These follow from Thm 2 and from the independence of observations of a random sample.