Ex 1: $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} N(\mu, 1)$ Let $n = 9$

$H_0: \mu = 0$
$H_1: \mu > 0$

(Testing simple null versus a composite alternative.)

Desired significance level is 5%.

a) Suppose RR (rejection region) has the form $RR = \{\bar{Y} > k_1\}$. Find k_1.

$$\alpha = 0.05 = P(\bar{Y} > k_1 \mid \mu = 0)$$
$$= P(X > k_1) \quad \text{where } X \sim N(0, 1/9)$$
$$= P\left(Z > \frac{k_1 - 0}{1/3}\right) = P(Z > 3k_1) = 0.05$$
$$\Rightarrow k_1 = \tfrac{1}{3}(1.645) = 0.548$$

b) Suppose RR has the form $RR = \{\bar{Y} < k_2\}$. Find k_2.

$$k_2 = -0.548$$

c) Compare the 2 RR's according to their power functions.

$RR_1 = \{\bar{Y} > 0.548\}$, $RR_2 = \{\bar{Y} < -0.548\}$

$$\text{Power}_1(\mu = 1) = P(\bar{Y} > 0.548 \mid \mu = 1)$$
$$= P\left(Z > \frac{0.548 - 1}{1/3}\right) = 0.9131$$

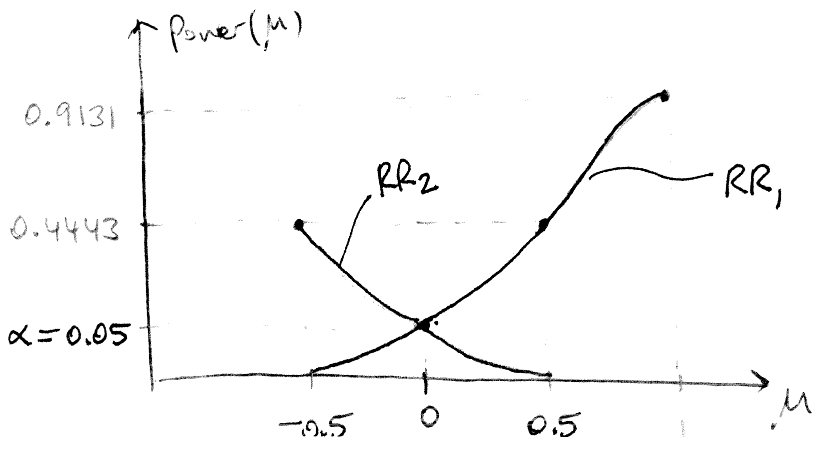
$\text{Power}_2(\mu = 1)$ is almost zero.

$\text{Power}_1(\mu = 0.5) = 0.4443$

$\text{Power}_1(\mu = -0.5) < 0.00135$

$\text{Power}_2(\mu = 0.5) < 0.00135$

$\text{Power}_2(\mu = -0.5) = 0.4443$



When building a new hypothesis test, we usually

1) fix α,
2) choose the RR to maximize power for the appropriate range of values.

In the previous example, RR_1 was preferable to RR_2. But, is $RR_1 = \{\bar{Y} > k_1\}$ the "best", in other words, the "most powerful"? Ideally, we would want power = zero at H_0 and power = 1 at H_1.

<u>Neymann-Pearson Lemma</u>:

Suppose we have a random sample of size n drawn from a pdf $f(\cdot)$.

Want to test $H_0: \theta = \theta_0$ versus $H_1: \theta = \theta_1$ with some <u>given</u> value of α.

The RR that has the most power at $\theta = \theta_1$ has the form

$$RR = \left\{ \frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\theta_1)} < k \right\}$$

<u>Idea</u>: If θ is more likely to be θ_1 than θ_0, $\mathcal{L}(\theta_1)$ will be large and $\mathcal{L}(\theta_0)$ will be small; so we will reject with more probability.

$\mathcal{L}(\theta)$ is the likelihood function.

Ex 2: $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$, σ_0^2 known.

$H_0: \mu = \mu_0$
$H_1: \mu = \mu_1$, $\mu_1 > \mu_0$

$$RR = \left\{ \frac{L(\mu_0)}{L(\mu_1)} < k \right\}$$

$$\frac{L(\mu_0)}{L(\mu_1)} = \frac{\frac{1}{(2\pi)^{n/2}} \frac{1}{(\sigma_0^2)^{n/2}} e^{-\frac{1}{2\sigma_0^2}\sum(y_i-\mu_0)^2}}{\frac{1}{(2\pi)^{n/2}} \frac{1}{(\sigma_0^2)^{n/2}} e^{-\frac{1}{2\sigma_0^2}\sum(y_i-\mu_1)^2}}$$

So, $RR = \left\{ k > \exp\left[-\frac{1}{2\sigma_0^2}\left(\sum(y_i-\mu_0)^2 - \sum(y_i-\mu_1)^2\right)\right] \right\}$

Take logs and rewrite,

$$RR = \left\{ -2\sigma_0^2 \ln k < -2(\mu_0-\mu_1)\sum y_i + n(\mu_0^2-\mu_1^2) \right\}$$

$$= \left\{ \frac{-2\sigma_0^2 \ln k - n(\mu_0^2-\mu_1^2)}{-2(\mu_0-\mu_1)} < \sum y_i \right\}$$

So, the RR has the form $\{\sum y_i > c\}$ or $\{\bar{Y} > d\}$
c, d : constants

$d = ?$

$$\alpha = P(\bar{Y} > d \mid \mu = \mu_0) = P\left(Z > \frac{d-\mu_0}{\sigma_0/\sqrt{n}}\right) = P(Z > z_\alpha)$$

$$\Rightarrow d = \mu_0 + z_\alpha \frac{\sigma_0}{\sqrt{n}}$$

Ex 3: $Y_1, Y_2, \dots, Y_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$

$H_0 : \mu = \mu_0$

$H_1 : \mu \neq \mu_0$

What is the most powerful test ?

There are infinite alternatives in $H_1 : \mu \neq \mu_0$!

i) Let's choose $\mu_1 = \mu_0 + 5$, i.e. $H_0 : \mu = \mu_0$
$H_1 : \mu = \mu_0 + 5$

$$RR = \left\{ \frac{\mathcal{L}(\mu_0)}{\mathcal{L}(\mu_0+5)} < k \right\} \quad \text{or} \quad \{ \bar{Y} > d \}$$

ii) Now consider $\mu_1 = \mu_0 + 10$.

$RR = \{ \bar{Y} > e \}$ $\quad$ e : a constant.

RR has the same form as above.

iii) Consider $\mu_1 = \mu_0 - 2$

$$RR = \{ \text{a constant} < -2\underbrace{(\mu_0 - \mu_1)}_{2} \textstyle\sum y_i \}$$

$$= \{ \text{a constant} < -4 \textstyle\sum y_i \} = \{ \bar{Y} < \underbrace{f}_{\text{a constant}} \}$$

RR has a different form here.

Neymann-Pearson lemma works when we have a single alternative.

What if we have many (an or infinite # of) alternatives ?

1) Select μ_1 such that $\mu_1 \neq \mu_0$.
2) Find the most powerful test of $H_0 : \mu = \mu_0$ versus
$H_1 : \mu = \mu_1$
3) Did you make any specific assumptions about μ_1 in step (2) ?

continued →

Suppose $H_0 : \mu = \mu_0$
$H_1 : \mu = \mu_1 \quad , \quad \mu_1 > \mu_0$

Is knowing only $\mu_1 > \mu_0$ enough to determine the form of the RR ?

If yes, then what you have found in step (2) is called a uniformly most powerful (UMP) test of the original hypothesis.

In Example 3, $H_0 : \mu = \mu_0$
$H_1 : \mu = \mu_1 \quad , \quad \mu_1 > \mu_0$

In this case, the form of RR is $\{ \bar{Y} > d \}$

This is a UMP test.

If $H_0 : \mu = \mu_0$
$H_1 : \mu = \mu_1 \quad , \quad \mu_1 < \mu_0 \quad \Rightarrow$ RR is $\{ \bar{Y} < e \}$

This is a UMP test.

* UMP does not always exist. For instance, it doesn't exist for double-sided alternative hypotheses.

One method of obtaining a RR is the Likelihood Ratio Test (LRT).

LRT's are not guaranteed to be UMP's.

But when a UMP test exists, the LRT is usually the same.

Suppose Ω is the space of θ, the unknown parameter.

$$\Omega = \Omega_0 \cup \Omega_1 \quad , \quad \Omega_0 \cap \Omega_1 = \phi$$

$H_0 : \theta \in \Omega_0$
$H_1 : \theta \in \Omega_1$

Then, the LRT has RR in the form

$$\left\{ \frac{\max_{\theta} \mathcal{L}_{\Omega_0}(\theta)}{\max_{\theta} \mathcal{L}_{\Omega}(\theta)} < k \right\}.$$

$\frac{\text{restricted maximum}}{\text{unrestricted maximum}}$

<u>Ex 4</u> : $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$, σ_0^2 known.

$H_0 : \mu = \mu_0$

$H_1 : \mu \neq \mu_0$

So, $\Omega_0 = \{\mu = \mu_0\}$

$\Omega_1 = \{\mu \neq \mu_0\}$

$$\mathcal{L}_{\Omega_0}(\mu) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{n/2} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(y_i-\mu_0)^2}$$

$\max_{\mu} \mathcal{L}_{\Omega_0}(\mu)$: maximized value of $\mathcal{L}_{\Omega_0}(\mu)$ over Ω_0 with respect to μ.

$\max_{\mu} \mathcal{L}_{\Omega_0}(\mu) = \mathcal{L}(\mu = \mu_0)$ since it is not a function of μ.

Now, check

$\max_{\mu} \mathcal{L}_{\Omega}(\mu)$: maximized value of the likelihood fnc. over the entire space Ω wrt μ.

We know that $\mathcal{L}(\mu)$ is maximized at $\hat{\mu}_{MLE} = \bar{Y}$

So, we have

$$\frac{\left(\frac{1}{2\pi\sigma_0^2}\right)^{n/2} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(y_i-\mu_0)^2}}{\left(\frac{1}{2\pi\sigma_0^2}\right)^{n/2} e^{-\frac{1}{2\sigma_0^2}\sum_{1}^{n}(y_i-\bar{y})^2}} =$$

$$= \exp\left[\frac{-1}{2\sigma_0^2}\left(\Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\bar{y})^2\right)\right]$$

Rewrite $(y_i-\bar{y})^2 = \left((y_i-\mu_0)-(\bar{y}-\mu_0)\right)^2$

$$\Sigma(y_i-\bar{y})^2 = \Sigma(y_i-\mu_0)^2 - n(\bar{y}-\mu_0)^2 \quad ---- \quad (*)$$

So,

$$RR = \left\{ \exp\left[-\frac{1}{2\sigma_0^2}\left(\Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\bar{y})^2\right)\right] < k \right\}$$

$$= \left\{ \ln\left[\exp[\cdot]\right] < \underbrace{\ln k}_{=k_1} \right\}$$

$$= \left\{ -\frac{1}{2\sigma_0^2}\left(\Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\bar{y})^2\right) < k_1 \right\}$$

$$= \left\{ \Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\bar{y})^2 < k_2 \right\}$$

using $(*)$

$$= \left\{ \Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\mu_0)^2 + n(\bar{y}-\mu_0)^2 < k_2 \right\}$$

$$= \left\{ n(\bar{y}-\mu_0)^2 < k_2 \right\}$$

$$= \left\{ (\bar{y}-\mu_0)^2 < k_3 \right\} = \left\{ |\bar{y}-\mu_0| < k_4 \right\}$$

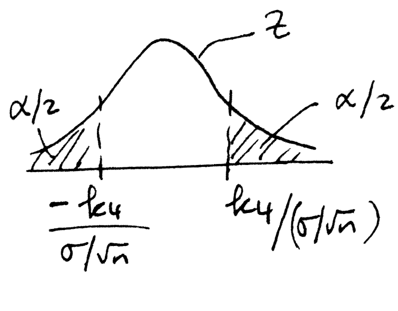
What is k_4?

Find k_4 so that the significance level of the test is α.

$$\alpha = P\left(|\bar{Y}-\mu_0| > k_4 \mid \mu=\mu_0\right) = P\left(|Z| > \frac{k_4}{\sigma/\sqrt{n}}\right)$$

$$\frac{k_4}{\sigma/\sqrt{n}} = z_{\alpha/2}$$

$$\Rightarrow k_4 = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$



So, the LRT has

$$RR = \left\{ \frac{|\bar{Y}-\mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2} \right\}$$

Ex 5 : $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$ μ unknown, σ^2 unknown

Test $H_0 : \mu = \mu_0$ versus
$H_1 : \mu \neq \mu_0$

Both hypotheses are composite (not simple) because we can not tell the distribution only by knowing μ_0. Even if we know μ_0, σ^2 is still unknown.

Here, $\theta = (\mu, \sigma^2)$

let $\tilde{\sigma}^2 = \frac{1}{n}\sum(y_i - \mu_0)^2$ (Need to estimate σ^2!)

Numerator

$$\max_{\mu, \sigma^2} \mathcal{L}_{\Omega_0}(\mu, \sigma^2) = \mathcal{L}(\mu_0, \tilde{\sigma}^2)$$

$$= \frac{1}{(2\pi)^{n/2}} \frac{1}{(\tilde{\sigma}^2)^{n/2}} e^{\underbrace{-\frac{1}{2\tilde{\sigma}^2}\sum(y_i-\mu_0)^2}_{(*)}}$$

$$(*) \quad \cdots \frac{1}{2\tilde{\sigma}^2}\sum(y_i-\mu_0)^2 = \frac{1}{2\frac{1}{n}\sum(y_i-\mu_0)^2}\sum(y_i-\mu_0)^2 = \frac{n}{2}$$

The numerator becomes : $\frac{1}{(2\pi)^{n/2}} \frac{1}{(\tilde{\sigma}^2)^{n/2}} e^{-n/2}$

Denominator

Unrestricted maximum.

$\max_{\mu, \sigma^2} \mathcal{L}(\mu, \sigma^2)$: This is equivalent to finding $\hat{\mu}_{MLE}$ and $\hat{\sigma}^2_{MLE}$.

We know that $\hat{\mu}_{MLE} = \bar{Y}$

$$\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(y_i - \bar{y})^2$$

The denominator becomes :

$$\frac{1}{(2\pi)^{n/2}} \frac{1}{(\hat{\sigma}^2)^{n/2}} e^{-\underbrace{\frac{1}{2\hat{\sigma}^2}\sum(y_i-\hat{\mu})^2}}$$

$$\frac{1}{2\frac{1}{n}\sum(y_i-\bar{y})^2}\sum(y_i-\hat{\mu})^2 = \frac{n}{2}$$

$\hat{\mu}=\bar{y}$

$$= \frac{1}{(2\pi)^{n/2}} \frac{1}{(\hat{\sigma}^2)^{n/2}} e^{-n/2}$$

So, LRT says :

$$RR = \left\{ \frac{\frac{1}{(2\pi)^{n/2}} \frac{1}{(\tilde{\sigma}^2)^{n/2}} e^{-n/2}}{\frac{1}{(2\pi)^{n/2}} \frac{1}{(\hat{\sigma}^2)^{n/2}} e^{-n/2}} < k \right\}$$

$$RR = \left\{ \left(\frac{\hat{\sigma}^2}{\tilde{\sigma}^2}\right)^{n/2} < k \right\} = \left\{ \frac{\hat{\sigma}^2}{\tilde{\sigma}^2} < k_1 \right\}$$

$$RR = \left\{ \frac{\frac{1}{n}\sum(y_i-\bar{y})^2}{\frac{1}{n}\sum(y_i-\mu_0)^2} < k_1 \right\}$$

$$= \left\{ \frac{\frac{1}{n}\sum(y_i-\bar{y})^2}{\frac{1}{n}\sum(y_i-\bar{y})^2 + n(\bar{y}-\mu_0)^2} < k_1 \right\}$$

Using $\frac{a}{a+b} = \frac{1}{1+\frac{b}{a}}$, $\quad \frac{1}{1+\frac{n(\bar{y}-\mu_0)^2}{\sum(y_i-\bar{y})^2}}$

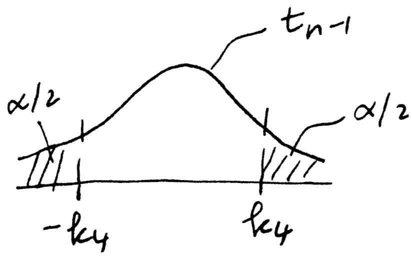
$$RR = \left\{ \frac{n(\bar{y}-\mu_0)^2}{\sum(y_i-\bar{y})^2} > k_2 \right\} = \left\{ \frac{n(\bar{y}-\mu_0)^2}{\frac{1}{n-1}\sum(y_i-\bar{y})^2} > k_3 \right\}$$

$$RR = \left\{ \frac{\sqrt{n}\, |\bar{y} - \mu_0|}{\left(\frac{1}{n-1} \sum (y_i - \bar{y})^2\right)^{1/2}} > k_4 \right\} = \left\{ \frac{\sqrt{n}\, |\bar{y} - \mu_0|}{\sqrt{s^2}} > k_4 \right\}$$

$k_4 = ?$

$$\alpha = P\left(\frac{|\bar{Y} - \mu_0|}{s/\sqrt{n}} > k_4 \;\middle|\; \mu = \mu_0 \right)$$

$\sim t_{n-1}$



$$k_4 = t_{n-1, \alpha/2}$$

Therefore,

$$RR = \left\{ \frac{\bar{Y} - \mu_0}{s/\sqrt{n}} > t_{n-1, \alpha/2} \right\}$$