Question 1: (2.5+ 2.5 = 5 points)

The time between customers at an ATM machine follows an exponential probability distribution with a mean time of 5 minutes between arrivals.

a) What is the probability that the time between two successive customers will be 3 minutes or less?
b) What is the probability that no customer arrives in a 4-minute time interval?

Solution:

a) Given that the time interval between customers is 5 minutes, customers per minute is $\lambda = 1/5$. $P(t < 3) = 1 - e^{-(1\,/\,5)*\,3}$
b) Poisson dist. with $\lambda = 4/5$ per 4 minutes. Then, $P(X = 0) = e^{-4/5}.\left(\frac{4}{5}\right)^0 / 0! = e^{-4/5}$.
Or, exponential: $P(t > 4) = 1 - [1 - e^{-(1/5)*\,4}]$.

Question 2: (5 points)

A delivery company divides their packages into weight classes. Suppose packages are uniformly distributed between 5 and 15 kilograms. What is the probability that a package will be between 8 and 10 kilograms?

Solution:

Uniform distribution. $f(X) = \frac{1}{15-5} = \frac{1}{10}$. Then, $P(8 < X < 10) = (10 - 8).\frac{1}{10} = \frac{2}{10}$.

Question 3: (6+6 = 12 points)

Suppose the heights of male university students are normally distributed. Suppose that a sample of 25 students has an average height of 170 centimeters. Build a 99% confidence interval around the population mean in the following two cases:

a) By assuming that the population variance is 25.
b) By assuming that the population variance is unknown and the sample variance is 25.

Solution:

a) Based on this sample, the 99% confidence interval for the population mean will be $\left(\bar{X} - Z_{a/2}\frac{\sigma}{\sqrt{n}},\right.$ $\left.\bar{X} + Z_{a/2}\frac{\sigma}{\sqrt{n}}\right) = \left(170 - 2.57\left(\frac{5}{5}\right),\ 170 - 2.57\left(\frac{5}{5}\right)\right) = (167.43\ ,172.57)$

b) $t_{n-1,a/2} = t_{24,0.005} = 2.797$.

Based on this sample, the 99% confidence interval for the population mean will be $\left(\bar{X} - t_{n-1,a/2}\frac{\sigma}{\sqrt{n}},\right.$ $\left.\bar{X} + t_{n-1,a/2}\frac{\sigma}{\sqrt{n}}\right) = \left(170 - 2.797\left(\frac{5}{5}\right),\ 170 - 2.797\left(\frac{5}{5}\right)\right) = (167.203\ ,172.797)$

Question 4: (6 points)

A random sample of 400 homes was taken from a large population of homes to estimate the proportion of homes with lead based paint. Out of 400 homes, 80 of them had lead based paint. Based on this information, build a 95% confidence interval for the true proportion of homes with lead based paint.

<u>Solution:</u>

We will assume that $\frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \sim N(0\,,1)$.

Therefore, the 95% CI will be $\left(\hat{p} - Z_{a/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + Z_{a/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$.

$$\left(0.20 - 1.96\sqrt{\frac{0.20(1-0.20)}{400}}, \qquad 0.20 + 1.96\sqrt{\frac{0.20(1-0.20)}{400}}\right)$$

$$= \left(0.20 - 1.96 * \frac{0.4}{20}, \qquad 0.20 + 1.96 * \frac{0.4}{20}\right) = (0.20 - 0.0392\,, 0.20 + 0.0392) = (0.1608, 0.2392)$$

Question 5: (6 points)

A random sample of size n = 25 is obtained from a normally distributed population with a population mean of 198 and a variance of 100. What is the probability that the sample mean is greater than 200?

$$P(\bar{X} > 200) = P\left(Z > \frac{200-198}{10/5}\right) = P\,(Z > 1) = 1 - P(Z < 1) = 1 - 0.8413 = 0.1587$$

Question 6: (6 points)

In a large city it was found that summer electricity bills for single-family homes followed a normal distribution with a standard deviation of $100. A random sample of 25 bills was taken. Find the probability that the sample standard deviation is less than $75.

$$P(S < 75) = P\left(\frac{(n-1)s^2}{\sigma^2} < \frac{24 *\ 75^2}{100^2}\right) = P(X_{24} < 13.5)$$

$0.95 < P(X_{24} > 13.5) < 0.975$

Then, $0.025 < P(X_{24} < 13.5) < 0.05$.

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Question 1: (5 points)

A delivery company divides their packages into weight classes. Suppose packages are uniformly distributed between 5 and 20 kilograms. What is the probability that a package will be between 10 and 15 kilograms?

Solution:

Uniform distribution. $f(X) = \frac{1}{20-5} = \frac{1}{15}$. Then, $P(10 < X < 15) = (15 - 10).\frac{1}{15} = \frac{5}{15}$.

Question 2: (6+6 = 12 points)

Suppose the heights of male university students are normally distributed. Suppose that a sample of 25 students has an average height of 170 centimeters. Build a 99% confidence interval around the population mean in the following two cases:

a) By assuming that the population variance is 36.
b) By assuming that the population variance is unknown and the sample variance is 36.

Solution:

a) Based on this sample, the 99% confidence interval for the population mean will be $\left(\bar{X} - Z_{a/2}\frac{\sigma}{\sqrt{n}},\ \bar{X} + Z_{a/2}\frac{\sigma}{\sqrt{n}}\right) = \left(170 - 2.57\left(\frac{6}{5}\right),\ 170 - 2.57\left(\frac{6}{5}\right)\right) = (166.916\ ,\ 173.084)$

b) $t_{n-1,a/2} = t_{24,0.005} = 2.797.$

Based on this sample, the 99% confidence interval for the population mean will be $\left(\bar{X} - t_{n-1,a/2}\frac{\sigma}{\sqrt{n}},\ \bar{X} + t_{n-1,a/2}\frac{\sigma}{\sqrt{n}}\right) = \left(170 - 2.797\left(\frac{6}{5}\right),\ 170 - 2.797\left(\frac{6}{5}\right)\right) = (166.6436\ , 173.3564)$

Question 3: (6 points)

A random sample of size n = 36 is obtained from a normally distributed population with a population mean of 198 and a variance of 100. What is the probability that the sample mean is greater than 200?

Solution:

$$P(\bar{X} > 200) = P\left(Z > \frac{200 - 198}{10/6}\right) = P\ (Z > 1.2) = 1 - P(Z < 1.2) = 1 - 0.8849$$
$$= 0.1151$$

Question 4: (6 points)

A random sample of 400 homes was taken from a large population of homes to estimate the proportion of homes with lead based paint. Out of 400 homes, 40 of them had lead based paint. Based on this information, build a 95% confidence interval for the true proportion of homes with lead based paint.

Solution:

We will assume that $\frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \sim N(0\,,1)$.

Therefore, the 95% CI will be $\left(\hat{p} - Z_{a/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + Z_{a/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$.

$$\left(0.10 - 1.96\sqrt{\frac{0.10(1-0.10)}{400}}, \quad 0.10 + 1.96\sqrt{\frac{0.10(1-0.10)}{400}}\right)$$

$$= \left(0.10 - 1.96 * \frac{0.3}{20}, \quad 0.10 + 1.96 * \frac{0.3}{20}\right) = (0.10 - 0.0294\,, 0.10 + 0.0294) = (0.0706, 0.1294)$$

Question 5: (6 points)

In a large city it was found that summer electricity bills for single-family homes followed a normal distribution with a standard deviation of \$100. A random sample of 23 bills was taken. Find the probability that the sample standard deviation is less than \$75.

Solution:

$$P\,(S < 75) = P\left(\frac{(n-1)s^2}{\sigma^2} < \frac{22\,.75^2}{100^2}\right) = P\,(X_{22} < 12.375).$$

$0.90 < P(X_{22} > 12.375) < 0.95$

Then, $0.05 < P(X_{22} < 12.375) < 0.10.$

Question 6: (2.5+ 2.5 = 5 points)

The time between customers at an ATM machine follows an exponential probability distribution with a mean time of 6 minutes between arrivals.

a) What is the probability that the time between two successive customers will be 4 minutes or less?
b) What is the probability that no customer arrives in a 5-minute time interval?

Solution:

a) Given that the time interval between customers is 5 minutes, customers per minute is $\lambda = 1/6$. $P(t < 4) = 1 - e^{-(1\,/\,6)*\,4}$

b) Poisson dist. with $\lambda = 5/6$ per 5 minutes. Then, $P(X = 0) = e^{-5/6}.\left(\frac{5}{6}\right)^0 / \,0! = e^{-5/6}$.

Or, exponential: $P(t > 5) = 1 - [1 - e^{-(1/6)*\,5}]$.