ECON 551 Quantitative Methods- Part 1: Probability and Statistics

<u>Homework 5</u>

1. Let $Y_1, Y_2, \dots, Y_n$ be a random sample from a normal distribution. In other words, $Y_1, Y_2, \dots, Y_n \sim iid\ N(\mu, \sigma^2)$. Define two alternative estimators for σ^2.

$\widehat{\sigma_1^2} = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ $\qquad$ $\widehat{\sigma_2^2} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$

a) Show that $\widehat{\sigma_1^2}$ is unbiased but $\widehat{\sigma_2^2}$ is biased.

b) Show that both $\widehat{\sigma_1^2}$ and $\widehat{\sigma_2^2}$ are consistent.

c) Compute the MSE for both estimators. Which estimator has a larger MSE?

2. $\bar{X}$ is the sample mean of a random sample of size n from distribution with mean μ and positive variance σ^2. Show that $\bar{X} \xrightarrow{p} \mu$ (converges in probability) if σ^2 is finite. (Hint: Use Chebychev's inequality.)

3. We showed in class that the distribution of the nth order statistic from a distribution with the pdf $f_X(x) = 1/\theta$ converges in distribution to a degenerate distribution function.

Now let $V_n = n(\theta - Y_n)$, where $Y_n$is the nth order statistic of the r.v X. Find the limiting distribution of V_n. In other words, show that the distribution of V_n converges in distribution to a cdf.

4. The Central Limit Theorem implies that the sample mean $\bar{X}_n$ of a random sample of size n i approximately $N(\mu, \sigma^2/n)$ for large n. Find the approximate distribution of $u(\bar{X}_n) = (\bar{X}_n)^3$.

5. Let $Y_1, Y_2, \dots, Y_n$ be a sequence of iid r.v.s. The common density of the r.v. is $f_Y(y;\theta) = (1/\theta)\exp(-y/\theta)$. Consider the following estimator for θ: $\hat{\theta} = cY_1 + (1-c)Y_2$ for some fixed c.
a) For which c is the estimator unbiased?

b) Calculate the variance of $\hat{\theta}$ as a function of c. Which value of c minimizes the variance among the set of unbiased estimators $\hat{\theta}$?

6. X has a binomial distribution with parameters 1 and 0.5. Y has a normal distribution with mean μ and variance 1. X and Y are independent. Consider an estimator $W = h(X, Y)$ for μ where $h(X, Y) = Y + 1 - 2X$.

a) What is the mean of W? $\qquad$ b) What is the variance of W?

c) Now consider the estimator $W_1 = \frac{1}{2}h(1, Y) + \frac{1}{2}h(0, Y)$. What are the mean and variance of W_1?

d) Is W_1 a better estimator for μ than W?

7. Let $Y_1, Y_2, \dots, Y_n$ be iid r.v.s drawn from $N(\mu, \sigma^2)$, both parameters are unknown. Find the ML estimators for μ and σ^2.

8. Let $X_1, X_2, \dots, X_n$ be iid r.v.s with pdf $f_X(x;\theta) = \frac{1}{\theta^2}x\, e^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$. Find the MLE for θ.